

Identifying noisy and quiet modes in a jet

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In the current jet noise study, an empirical modal decomposition is proposed which distills the noisy and quiet modes of the flow field. In particular, the POD of flows is generalised for an optimal resolution of the far-field noise as opposed to a least-order representation of the hydrodynamic fluctuation level. This decomposition technique, which we call ‘most observable decomposition (MOD)’, is based on a linear cause-effect relationship between the hydrodynamics (cause) and the far-field acoustics (observed effect). In the current study, this relationship is identified from a linear stochastic estimation between the flow field and the far-field pressure — taking into account the propagation time of sound. We employ MOD to turbulent jet noise at $Ma = 0.9$, $Re = 3600$ using CFD/CAA data from RWTH Aachen. While more than 350 POD modes are necessary to capture only 50% of the flow fluctuation energy, a mere 24 MOD modes resolve 90% of the far-field acoustics. Evidently, far-field noise acts as filter which ‘sees’ only a low-dimensional subspace of the flow and ‘ignores’ silent subspaces which contain a large amount of fluctuation energy. The MOD methodology yields ‘least-order’ representations of any other observable as well — assuming a linear relationship between flow and observable.

I. Introduction

At present there is no general consensus as to precisely which aspects of a free jet’s flow dynamic are most important in the production of sound. There is of course a large body of experimental and numerical evidence for a number of ‘hot-spots’ in terms of sound production, for example: the close of the annular mixing-layer at the end of the potential core and the subsequent transition to self-similarity have been argued to be important by a number of researchers,^{1,2,3,4,5} violent intermittent entrainment of coherent structures into the high-velocity potential regions of the flow—also interpreted as the collapse of large-scale instabilities^{13,8}—being shown to be important; *quasi*-irrotational wavy-wall-like mechanisms, probably most important in the region upstream of the close of the potential core;⁶ vortex pairing, again in regions upstream of the end of the potential core;⁷ vortex eigen-oscillations associated with coherent structures;^{10,11,12} incoherent fine-scale turbulence;⁹ instabilities in the near-nozzle region, possibly rendered efficient by the presence of the nozzle.¹⁴ However, the precise nature and relative contributions of each of these mechanisms are not well understood.

In this work a methodology is developed, based on modal decompositions of the flow and acoustic fields of a $M=0.9$, $Re=3600$ jet computed by an LES of the flow and CAA, based on APE and FWH formulae.^{15,16} This methodology aims to identify the most important flow events with respect to a number of observer points in the acoustic far-field. The technique involves monitoring the flow over linearly related observables measured in an L_2 norm. The least dimensional flow representation is then derived with respect to an observable subspace derived from the far-field pressure.

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II. Outline of the technique

Proper Orthogonal Decomposition (hereafter POD) is first used to obtain a Galerkin approximation of the flow field:

$$\mathbf{u}^{[0..N]}(\mathbf{x}, t) := \sum_{i=0}^N a_i^u(t) \mathbf{u}_i(\mathbf{x}), \quad (1)$$

where the mean field is termed \mathbf{u}_0 ($a_0^u = 1$) and the POD modes \mathbf{u}_i are spatial expansion modes. The POD modes are obtained by the POD snapshot method, which is based on a snapshot ensemble $\{\mathbf{u}^i\}_{i=1}^{M_u}$ of the flow field utilising the above mentioned LES. Two key elements in POD snapshot method¹⁷ are: the inner product of the velocity fluctuations,

$$(\mathbf{u}, \mathbf{v})_u := \int d\mathbf{x} \mathbf{u}(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x}), \quad (2)$$

and the correlation matrix,

$$R_u^{mn} = \frac{1}{M_u} (\mathbf{u}^m - \mathbf{u}_0, \mathbf{u}^n - \mathbf{u}_0)_u, \quad (3)$$

which comprises second-order structural information of the turbulence. The POD modes \mathbf{u}_i decompose the flow field most efficiently for total kinetic energy (total fluctuation level $\frac{1}{M_u} \sum_{m=1}^{M_u} R_u^{mm}$), and are obtained by means of a diagonalisation of the correlation matrix; and the inner-product determines the optimality of the modal decomposition. This is an important point which forms the basis of the technique presented in this work. It is worth noting that this first step does not necessarily need to be restricted to the velocity field: a flow quantity such as an acoustic-analogy source term can alternatively be used, in which case a Galerkin approximation of the source dynamics is obtained.²⁵ Where the approach described in this paper is concerned such a choice may present considerable advantages, as the relationship between such source terms (Lighthill for example) and the radiated sound pressure is generally a linear one. The downside of such an approach is that an evolution equation is not readily available for acoustic-analogy source-terms, and thus a straight forward low-order dynamical-system identification (LODS) is not feasible.

The radiated pressure field, sampled on a line array at 30 diameters from and parallel to the jet axis, is similarly represented by its Galerkin approximation:

$$\mathbf{p}^{[0..N_p]}(\mathbf{x}, t) := \sum_{i=0}^{N_p} a_i^p(t) \mathbf{p}_i(\mathbf{x}), \quad (4)$$

which is also obtained by POD snapshot method, using the inner product

$$(p, q)_p := \int dx p(r, \theta, x) q(r, \theta, x) \Big|_{r=30D, \theta=0^\circ}, \quad (5)$$

and the correlation matrix

$$R_p^{mn} = \frac{1}{M_p} (p^m - p_0, p^n - p_0)_p, \quad (6)$$

based on the far-field snapshot ensemble $\{p^i\}_{i=1}^{M_p}$. Here, the POD modes p_i decompose the far-field most efficiently for the total level of far-field fluctuation $\frac{1}{M_p} \sum_{m=1}^{M_p} R_p^{mm}$.

The highly-complex dynamics of the flow, and the somewhat less-complex dynamics of the acoustic pressure field are thus characterised by the POD modes $\mathbf{u}_i(\mathbf{x})$ and $\mathbf{p}_i(\mathbf{x})$, and the Fourier coefficients $a_i^u(t)$ and $a_i^p(t)$: the Fourier coefficients animate the eigenfunctions. The POD is shown schematically in figure 1.

We wish to understand the causal relationship which translates the physical mechanisms by which the flow dynamics ($\sum_{i=0}^N a_i^u(t) \mathbf{u}_i(\mathbf{x})$) drive the far-field ($\sum_{i=0}^{N_p} a_i^p(t) \mathbf{p}_i(\mathbf{x})$). In a first stage we assume a linear relationship between the fluctuating turbulent velocity and the acoustic far-field pressure: there is a considerable

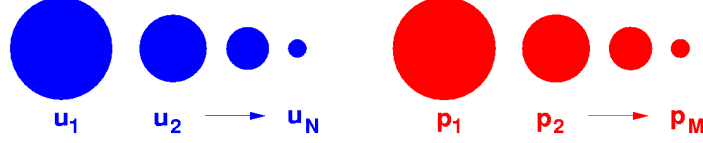


Figure 1. Schematic representation of POD modes: size of circles represents the level of resolved fluctuation level of each mode; in blue: velocity modes, optimal resolution of total kinetic energy based on the inner-product $(u, v)_u$; in red: pressure modes, optimal resolution of total far-field fluctuation level based on the inner-product $(p, q)_p$.

body of evidence to suggest that such an approximation is reasonable: the fast-pressure term (sometimes referred to as ‘shear-noise’) has been shown to dominate in free-jets in terms of the hydrodynamic, turbulent pressures,²⁶ and to correlate better with the far-field pressure than the quadratic slow-pressure (‘self-noise’);^{18, 21, 24, 22, 23, 19, 1, 20, 29} Coiffet et al.⁶ have furthermore demonstrated that the coherent flow structures generate sound via a linear mechanism in the region upstream of the end of the potential core, and Guitton et al.³⁰ have shown that such a mechanism exists as far downstream as 10D, and for Mach number ranging from 0.15 to 0.6: *we therefore perform a linear mapping between the POD mode coefficients of the velocity field and those of the pressure field:*

$$a_i^p(t) = \sum_{j=1}^N C_{ij}^{pu} a_j^u(t - \tau), \quad (7)$$

where τ represents the time delay of acoustic propagation and C_{ij}^{pu} —which embodies the degree of linearity between the dynamics of the turbulent velocity field and the acoustic far-field (via their Fourier coefficients)—is obtained using Linear Stochastic Estimation.

The role of the operator \mathbf{C} , given by the matrix elements C_{ij}^{pu} , can be understood in a number of ways: it provides a means of stripping the highly complex turbulence dynamics down to the components which were essential in linearly driving the acoustic far-field; as such, it amounts to a filtering operation which identifies the radiating jet dynamics (very similar to the action of a Green’s function): convolving \mathbf{C} with the velocity field (a^u) gives the far-field pressure; convolving its pseudo-inverse, \mathbf{C}^- , with the pressure field (a^p) identifies the turbulence dynamics which were linearly implicated in the production of sound, or in other words, linearly observable from the far-field (see figure 2: blue circles represent the observable jet dynamics); it allows us to study the jet dynamics in a linear observer sub-space which has considerably fewer degrees of freedom than the complete turbulence field. These ideas are illustrated schematically in figures 2. For a particular pseudo-inverse, which minimises the energy of the image, this procedure coincides with the method of extended POD.^{31, 32}

In figure 3 the blue phase-diagrams represent (very simplified) model jet dynamics, in two and three dimensions; the green lines represent linear mappings, which define the projections of the higher-dimensional jet dynamics onto the linear subspaces (shown in red), as seen by a single observer in the 2D case, and by two observers in the 3D case. In order to discern the complexity of the dynamics in the linear subspace a Proper Orthogonal Decomposition could be performed; however, in such a situation, because the inner product of the problem is defined using the velocity field, such a decomposition is optimal in terms of the kinetic energy of the filtered velocity field. *We are interested in a decomposition which is optimal in terms of the sound energy radiated by the flow*, and so we redefine the inner product as follows:

$$(\mathbf{u}, \mathbf{v})_* := (\mathbf{C} \mathbf{u}, \mathbf{C} \mathbf{v})_p = \sum_{i=1}^S \left(\sum_{j=1}^N C_{ij}^{pu} a_j^u \right) \left(\sum_{j=1}^N C_{ij}^{pu} a_j^v \right). \quad (8)$$

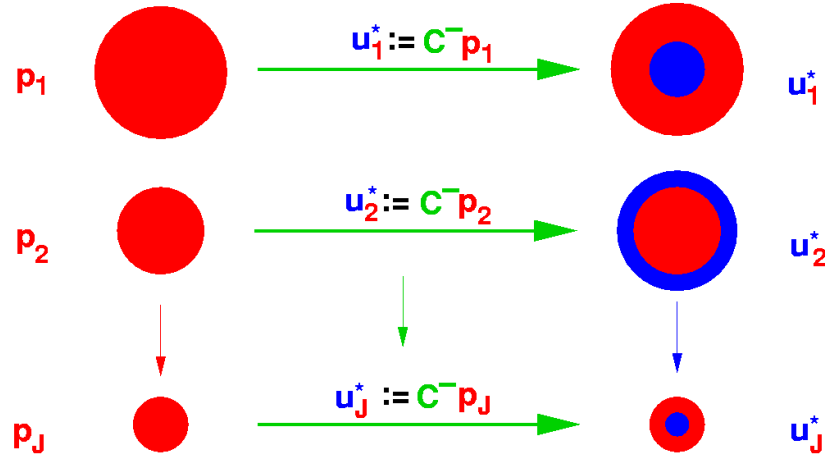


Figure 2. Schematic representation of MOD: acoustically optimised Galerkin representation of jet's 'causal' linear-subspace. These linear subspaces are determined to be the pseudo-inverse images of the far-field POD mode. Blue and red circles are explained like in figure 1. The property of optimal far-field fluctuation resolution is maintained for a sufficiently accurate cause-effect relationship (7) of hydrodynamics and aeroacoustics.

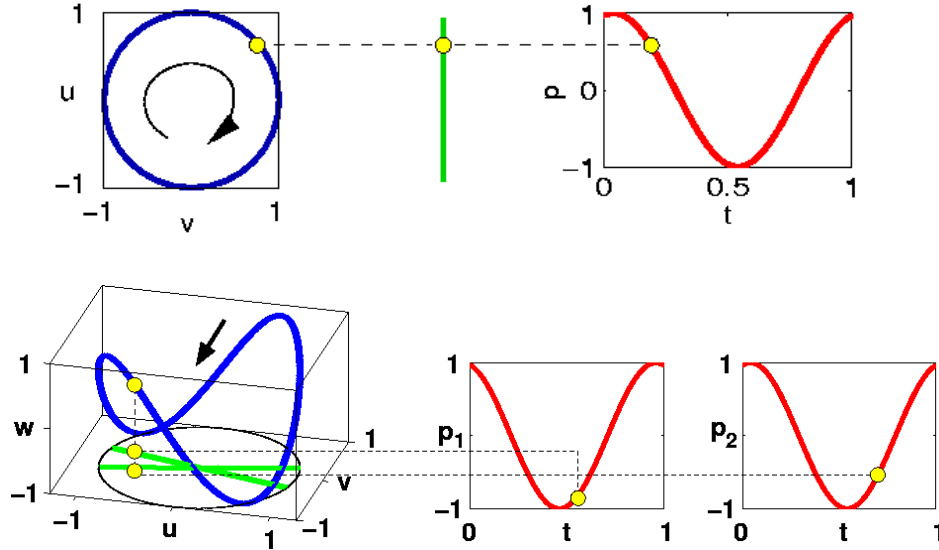


Figure 3. Two- and three-dimensional schematic representations of the link (via linear mapping: shown in green) between complex high-dimensional jet dynamics (in blue) and the less-complex, lower-dimensional acoustic dynamics (in red). In the two-dimensional representation on the top row, the (red) far-field signal $p = \cos 2\pi t$ ($t \in [0, 1]$) can be observed by the projection $p = C u$ of the (blue) field $u = (\cos 2\pi t, \sin 2\pi t)^T$ onto the linear subspace $C = (1, 0)$ (green line). In the three dimensional representation, the (red) far-field signals $p_1 = \sqrt{\frac{15}{16}} \cos 2\pi t - \frac{1}{4} \sin 2\pi t$ and $p_2 = \sqrt{\frac{15}{16}} \cos 2\pi t + \frac{1}{4} \sin 2\pi t$ can be observed by the projection $(p_1, p_2)^T = C u$ of the (blue) field $u = (\cos 2\pi t, \sin 2\pi t, \sin 4\pi t)^T$ onto the linear subspaces represented by the rows of $C = (C_1, C_2)^T$ with $C_1 = \left(\sqrt{\frac{15}{16}}, -\frac{1}{4}\right)$ and $C_2 = \left(\sqrt{\frac{15}{16}}, \frac{1}{4}\right)$ respectively (green lines).

The POD problem is thus posed such that: (1) the kernel of the problem comprises two-point correlations of the filtered velocity field, and so velocity modes will be obtained; but, (2) *the modal decomposition is optimal with respect to the sound energy radiated from the flow*. This is the MOD principle, and it is shown schematically in figure 4.

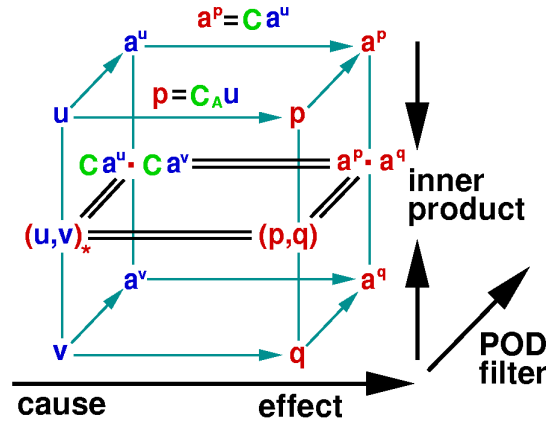


Figure 4. Schematic representation of MOD idea: cause (velocity, blue) and effect (pressure, red): left and right-hand vertical faces of cube; POD filter (Galerkin representations of dynamics): front and back vertical faces of cube; cause and effect related by linear-mapping (green): this is the key to identifying the jet’s ‘causal’ subspace; identity between inner-products (middle horizontal section) is the key to identifying loud ‘causal’ modes, i.e. MOD modes. This figure is explained in more detail below.

The left and right faces of the cube represent, respectively, the causal jet dynamics in terms of its velocity field $\mathbf{u}(\mathbf{v})$ and their effect, the far-field pressure $\mathbf{p}(\mathbf{q})$: they are related by the linear mapping \mathbf{C} . Moving from the front face of the cube to the back face we change from a classical space-time description of the two fields to Galerkin approximations of these by means of a POD filter (dynamics now captured by the Fourier coefficients $\mathbf{a}^u, \mathbf{a}^v, \mathbf{a}^p$ and \mathbf{a}^q). The central horizontal plane contains the inner-products, and it is here that we find the key to the MOD: *a velocity inner-product is defined which is equal to the inner-product of the far-field pressure*: this forces an optimisation of the jet dynamics—which are here filtered by the linear mapping—in terms of the far-field pressure when the POD filter is performed, and allows thence an identification of the most linearly observable velocity modes.

III. Results

The first important result is manifest in the MOD convergence: this is shown in figure 5, where it is compared with the convergence of the POD. While more than 350 POD modes are necessary to capture 50% of the TKE of the jet, 90% of the radiated sound energy is captured with only 24 MOD modes. This is a testament to the significantly lower-dimensional nature of the sound-producing jet dynamics: such reductions in complexity were also observed, respectively, by Freund²⁷ and by Cabana et al.,²⁸ when they isolated the radiating component of the Lighthill source term: in a low Reynolds-number jet (very similar to the one studied here), and in a two-dimensional temporal mixing-layer. It should be emphasised that this reduced complexity of the sound-producing jet dynamics is good news for control: while the complexity of a turbulent jet presently precludes the direct application of control theory, the existence of such a low-order sound-producing component will be conducive to the identification of a dynamical-system, a dynamic observer and an efficient control law.

A second interesting result is manifest—with respect to the physical mechanisms which underlie the production of sound in this low Reynolds-number flow—in the first six MOD modes, shown in figure 6. The higher MOD modes all show variously disorganised, smaller-scale activity and are not shown here. It can

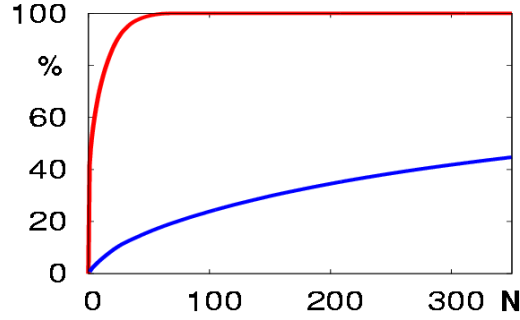


Figure 5. Convergence of POD and MOD: The resolution of total kinetic energy by N POD modes is displayed in percentages over N (blue line). The resolution of total far-field fluctuation level by N MOD modes is displayed in percentages over N (red line).

be seen how the MOD modes translate a number of different characteristic flow features and events. The first two, which are the most energetic modes in terms of the far-field pressure (resolving about 48% of the far-field), reveal asymmetric (helical-looking) streaks in the region just downstream of the end of the potential core. This is in qualitative agreement with experiments such as those of Juve *et al.*,¹ Guj *et al.*,² and Hileman *et al.*⁴ The next three most energetic modes (which contribute about 12% of the far-field) show structures comprising highly coherent, axisymmetric vortex-ring-like structures in the region upstream of the end of the potential core. This resembles the wavy-structure of the radiating component of the Lighthill source term, as identified by Freund,²⁷ and is in excellent agreement with the linear wavy-wall mechanism identified experimentally by Coiffet *et al.*⁶ While the higher order modes (which account for 40% of the far-field) reveal a more disorganised structure, mainly downstream of the end of the potential core, they remain relatively structured: mode six shows the remnants of the axisymmetric flow-structures, particularly in the v-component.

In summary: the sound producing jet dynamics — as identified by MOD in this low Reynolds-number flow—are dominated by convected axisymmetric vortex-rings in the region upstream of the end of the potential core. Their non-negligible contribution to the far-field (12%) is an interesting result; and one which is contrary to: the school-of-thought which holds that this mechanism is inefficient on account of its subsonic convection velocity; and, the argument that such mechanisms require non-linear mode interactions to become efficient in the production of sound. (remember a linear relationship has been assumed here between the velocity fluctuations and the far-field; a similar linear mechanisms is demonstrated by Coiffet *et al.*,⁶ and, another strongly linear coupling between hydrodynamic pressure and velocity is proved by Noack *et al.*³³). At the end of the potential core these axisymmetric structures are destroyed and transformed into streak-like structures (visual inspection reveals an azimuthal mode 5 or 6): this is the dominant sound-producing event in this flow, contributing of the order of 48%, and it is in agreement with a large body of experimental and numerical evidence.

IV. Conclusion

A method which we call the Most Observable Decomposition (MOD) has been developed for identifying the sound-producing dynamics of a turbulent jet. Application to numerically computed low Reynolds number jet has led to the following observations: (1) while over 350 POD modes are required to capture 50% of the TKE, 90% of the sound energy is capture with only 24 MOD; (2) the MOD mode demonstrate how over 48% of the sound generated by this flow is due to the destruction of coherent structures in the transition region, downstream of the end of the potential core; (3) over 12% of the sound production is found to be

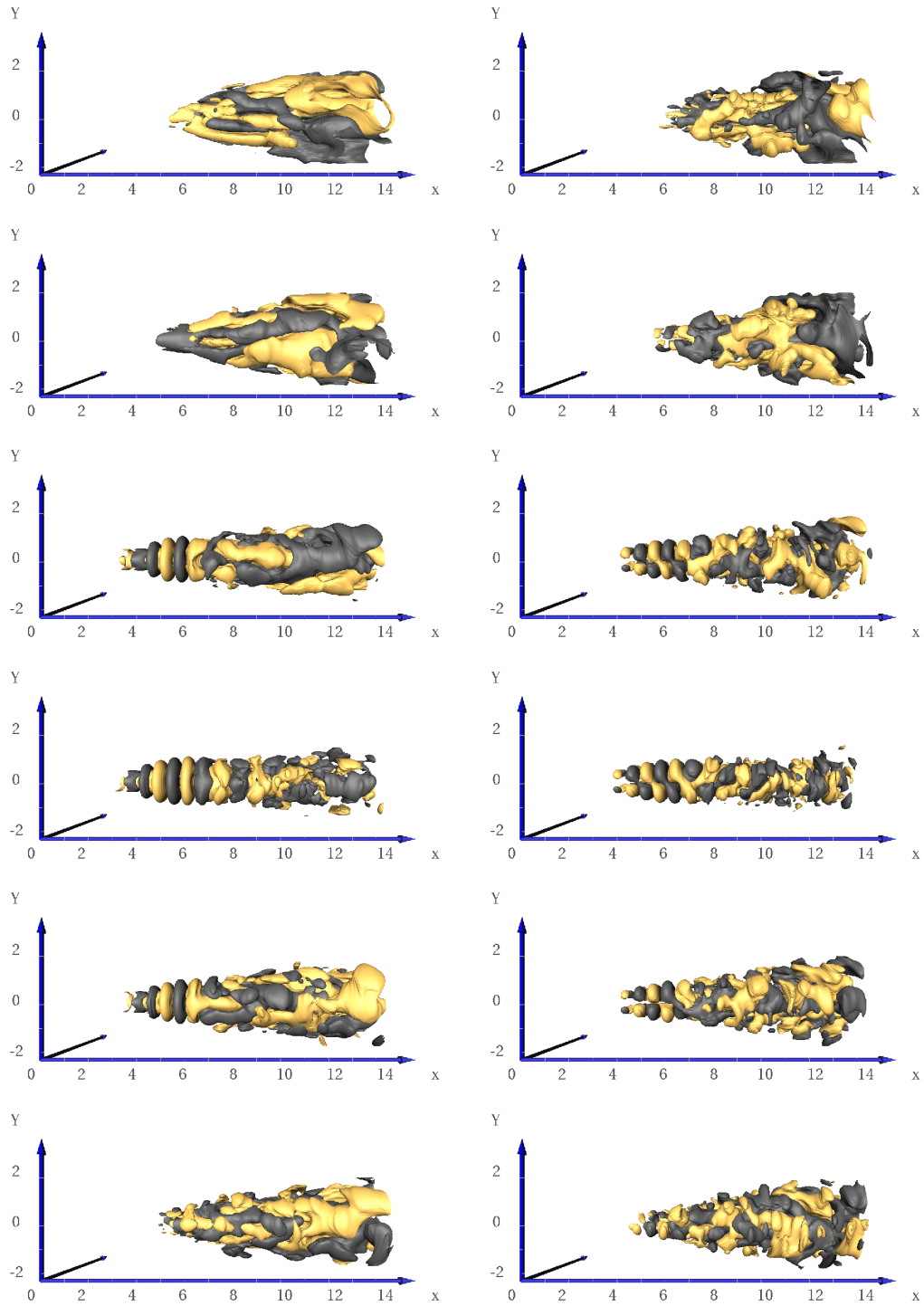


Figure 6. First six (1 to 6 from top to bottom) 3D MOD modes: left column: u -component; right column: radial component. Isosurfaces of positive (yellow) and negative (dark) values are shown.

related to convected axisymmetric vortex-rings, whose signature persists into the transition region. Further analysis of this data is ongoing

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