Low-dimensional modelling — Motivation and introduction

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Berlin Institute of Technology

CISM course "Reduced-Order Modelling for Flow Control", Udine, September 15–19, 2008

My lectures

1 (Mo) Motivation of Galerkin method, 2 examples

Purpose of this lecture:

- Illustrate the need for reduced-order models (ROM)
- Introduce the Galerkin method.

2 (Tu) Empirical Galerkin method based on POD

3 (Tu) POD-based Galerkin models of natural flow

4 (Th) POD-based Galerkin models of transient and actuated flow

5 (Th) Towards an attractor control

Overview

1. Navier-Stokes equation and flow phenomenology
2. Motivation for reduced-order models
3. Examples of Galerkin models
4. Take home messages

Navier-Stokes equation

Incompressible flow: $\mathbf{x} = (x, y, z), t \mapsto \mathbf{u} = (u, v, w), p$

Normalization: U: characteristic velocity
D: characteristic size
ρ: density

Reynolds number: $Re = UD/\nu$, ν= kinematic viscosity

Mass conservation: $\nabla \cdot \mathbf{u} = 0$

Momentum balance:

$$\underbrace{\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}}_{\text{acceleration}} = \underbrace{-\nabla p}_{\text{pressure force}} + \underbrace{\frac{1}{Re} \triangle \mathbf{u}}_{\text{viscous force}}$$

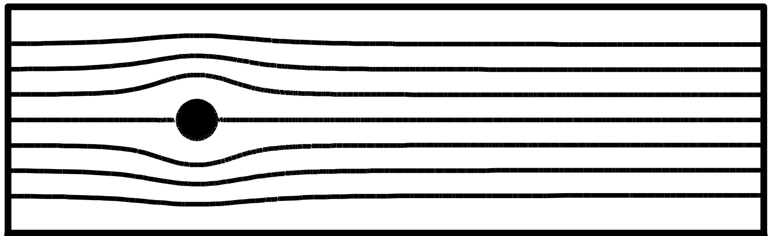
Solution:

- Flow has a single attractor
- Turbulence attractor $N \sim Re^{9/4}$, e.g., $Re = 10^6 \Rightarrow N \sim 3 \cdot 10^{13}$

Phenomenogram of cylinder wake

Reynolds number $Re = \frac{UD}{\nu}$

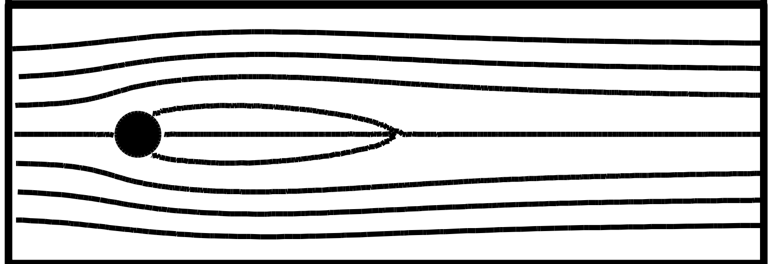
$Re < 4$



2D steady flow

without vortex pair

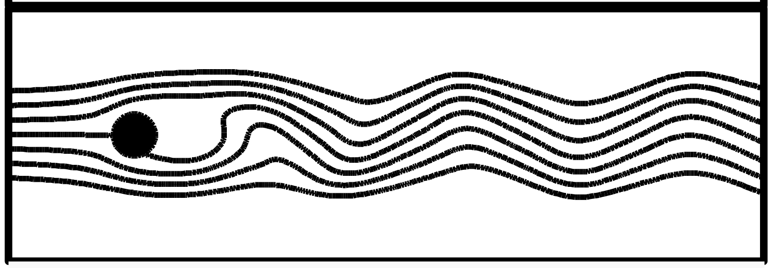
$Re < 47$



2D steady flow

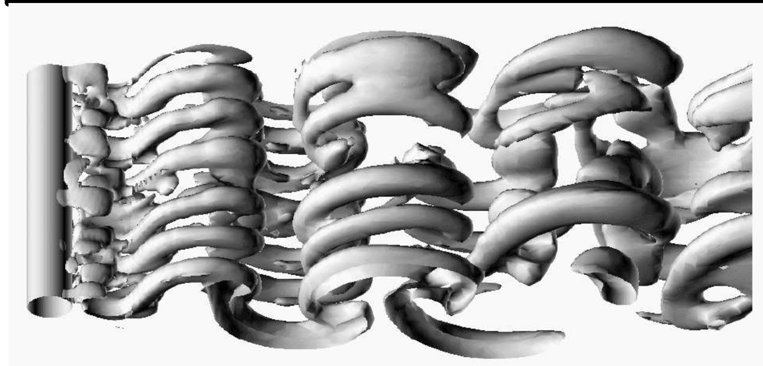
with vortex pair

$Re < 180$



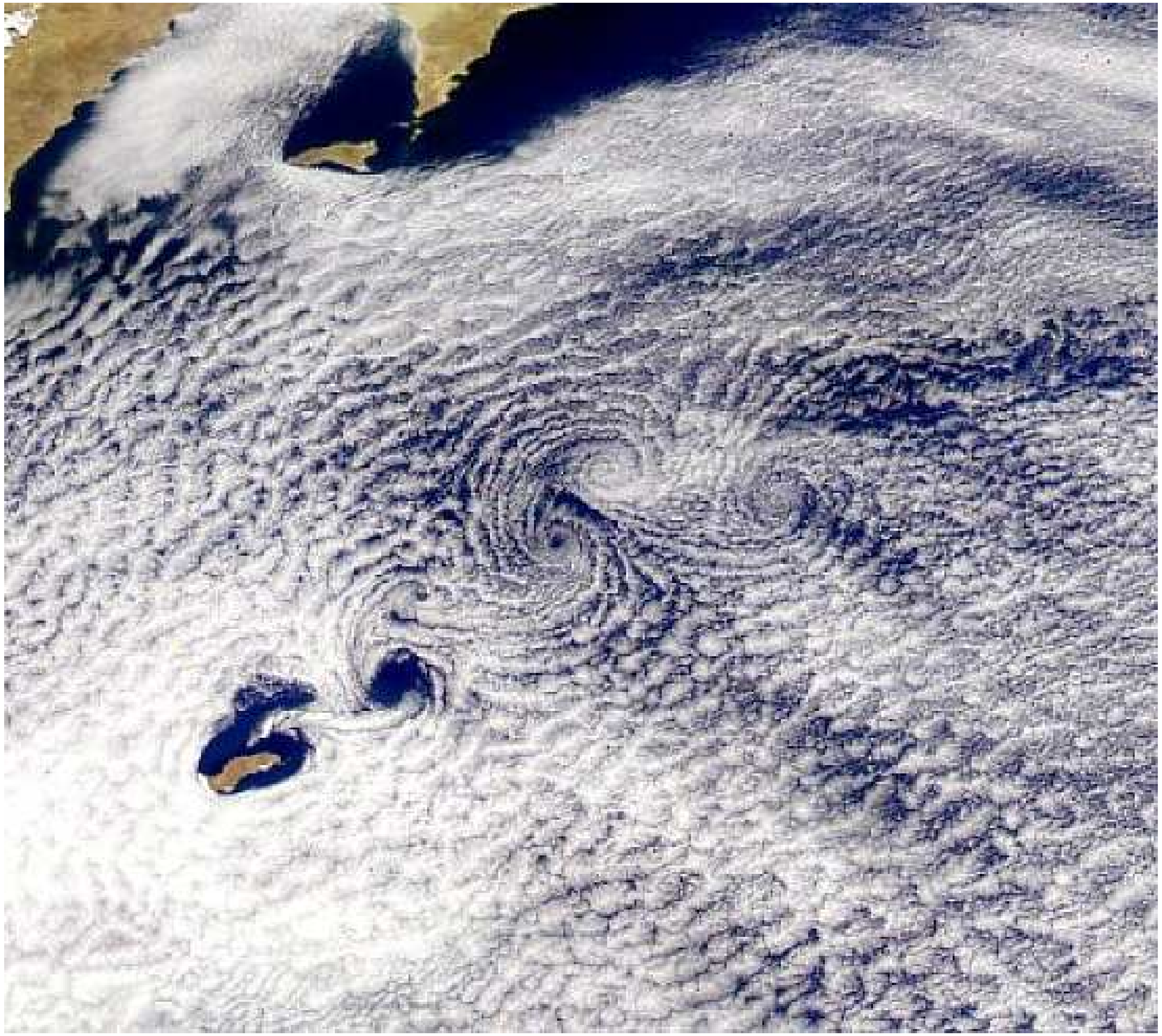
2D vortex shedding

$180 < Re$



2D vortex shedding superimposed by 3D modes / fluctuations

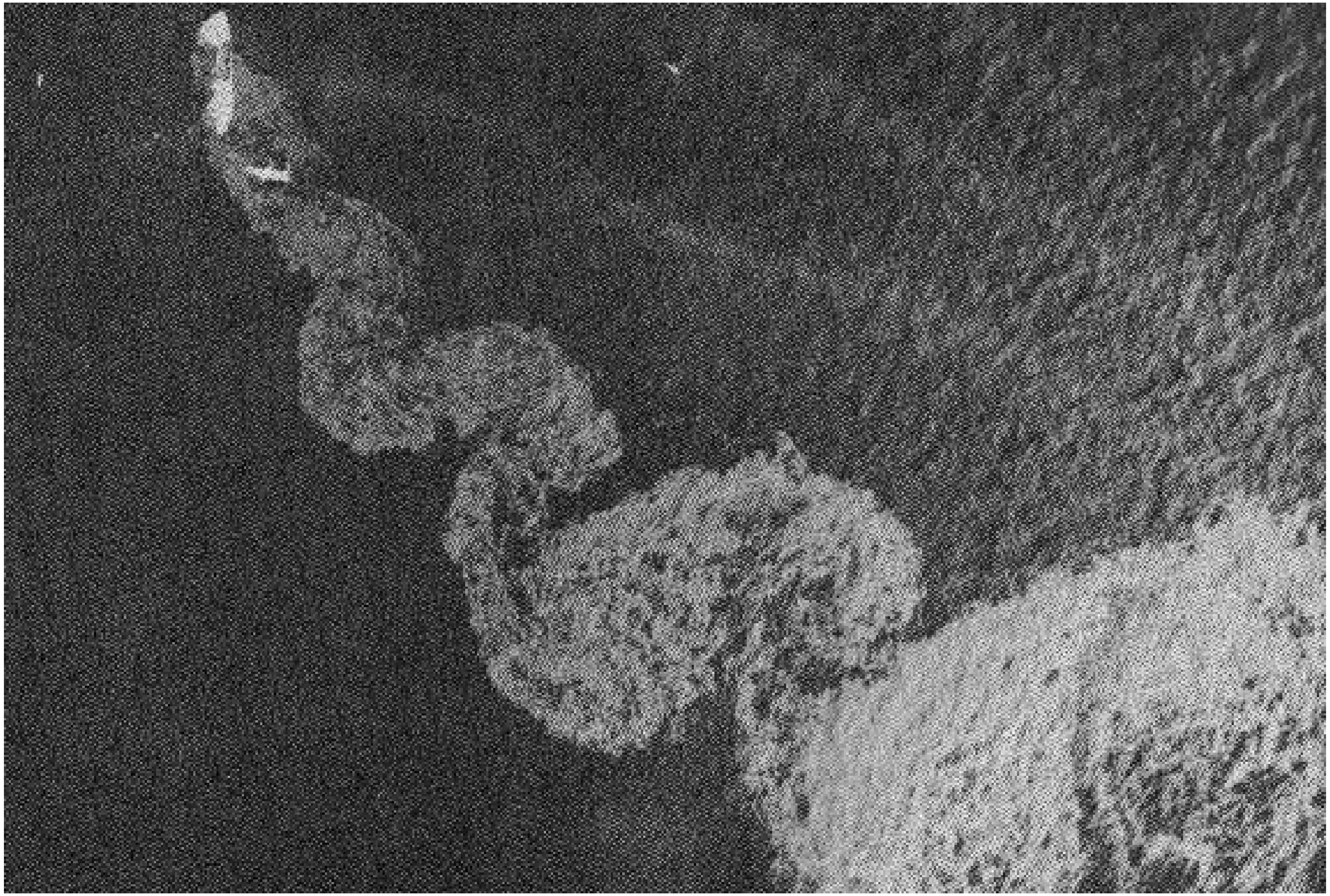
von Kármán vortex street in nature



Rear side of the island Guadalupe (20 Aug. 1999)

von Kármán vortex street in technology

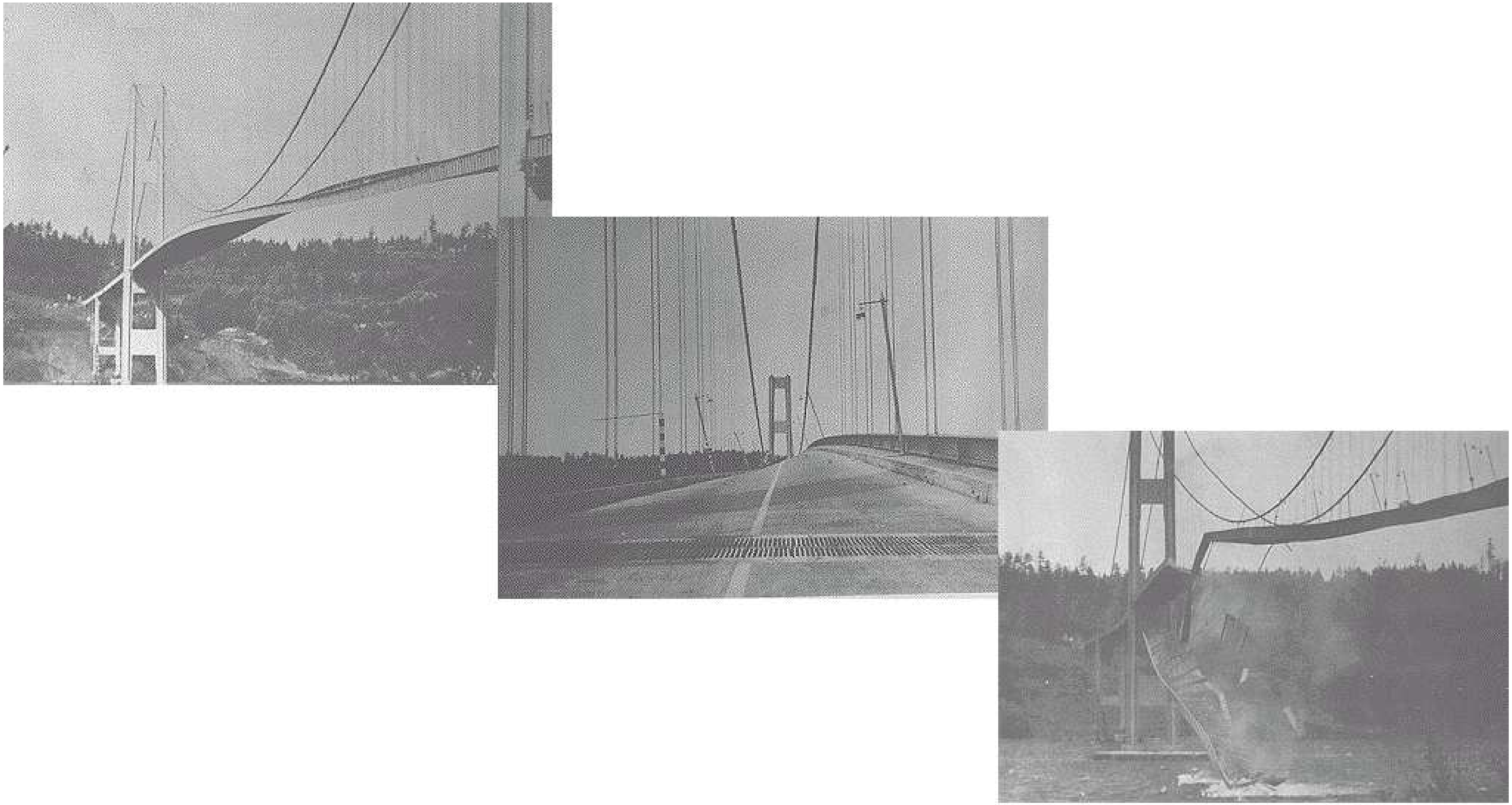
Damaged tanker — oil visualization



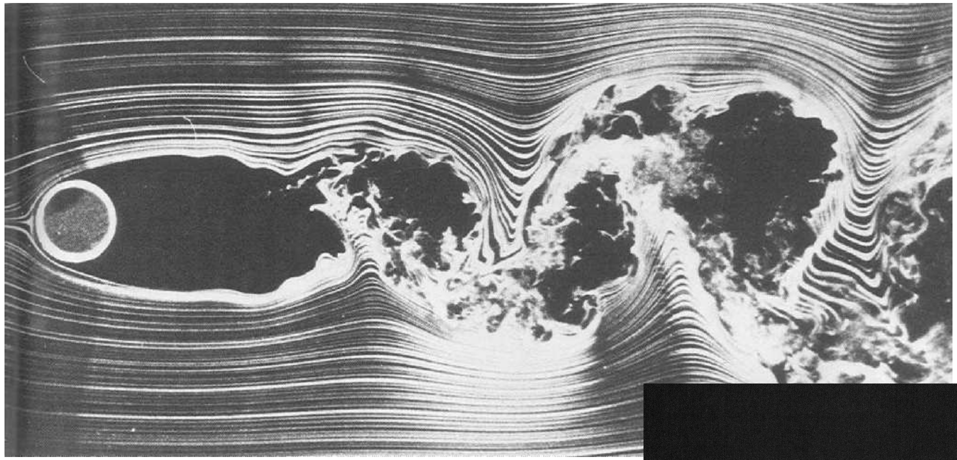
Van Dyke (1982), page 100

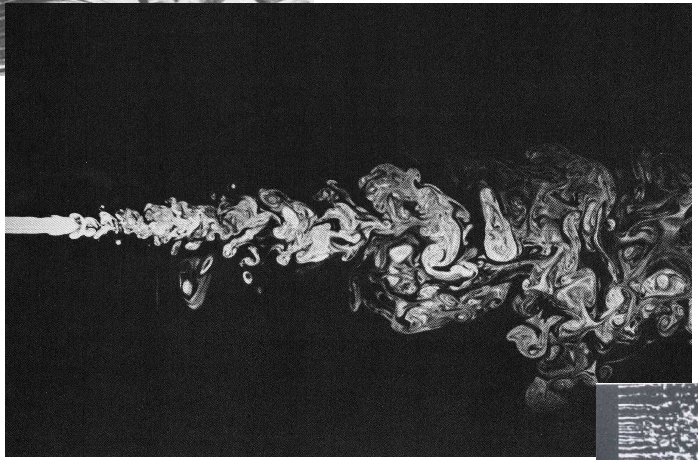
von Kármán vortex street in technology

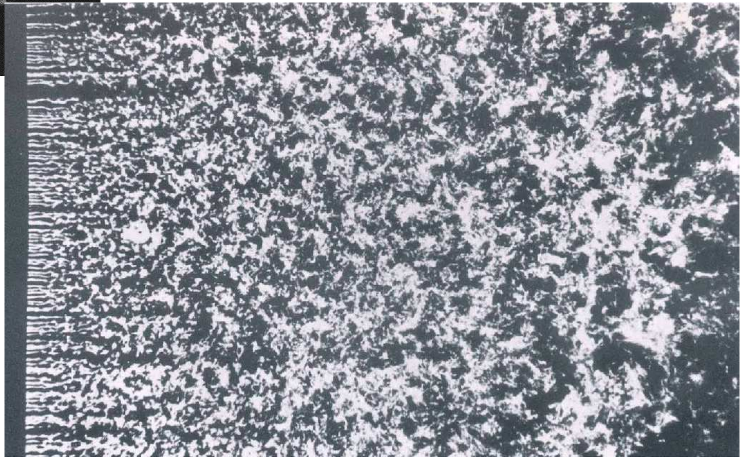
Tacoma Narrows Bridge (7 Nov. 1940)



Flow phenomenologies

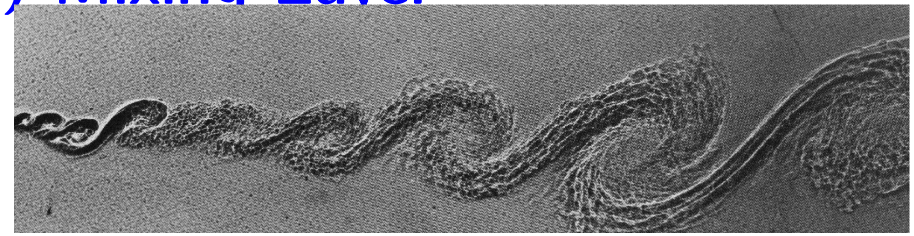
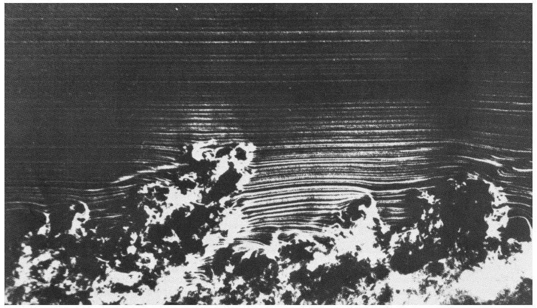
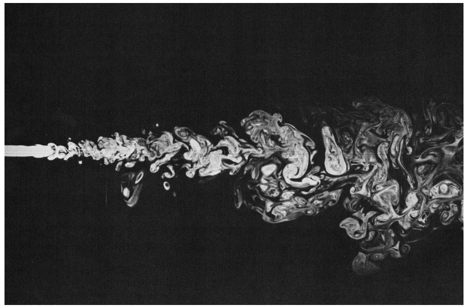
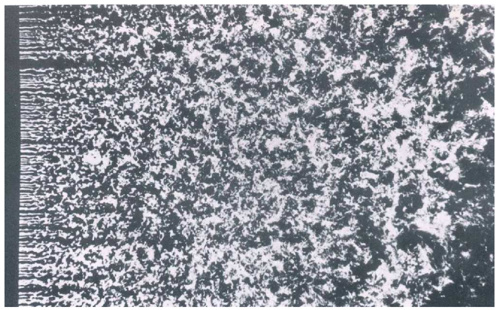
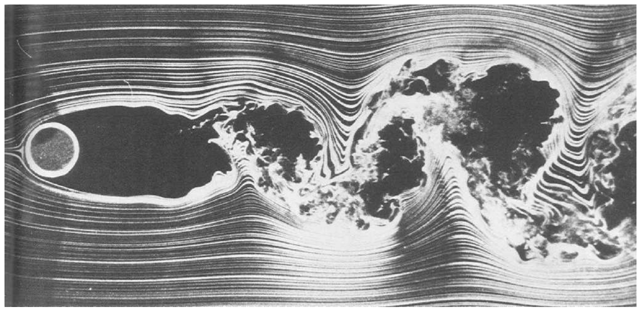






Milton van Dyke (1975): Album of Fluid Motion

Göttingen shear flow classification

wall-bounded	free
1) Couette/channel flow 	3) Mixing Layer 
2) Boundary layer 	4) Jet 
6) Isotropic turbulence 	5) Wake 

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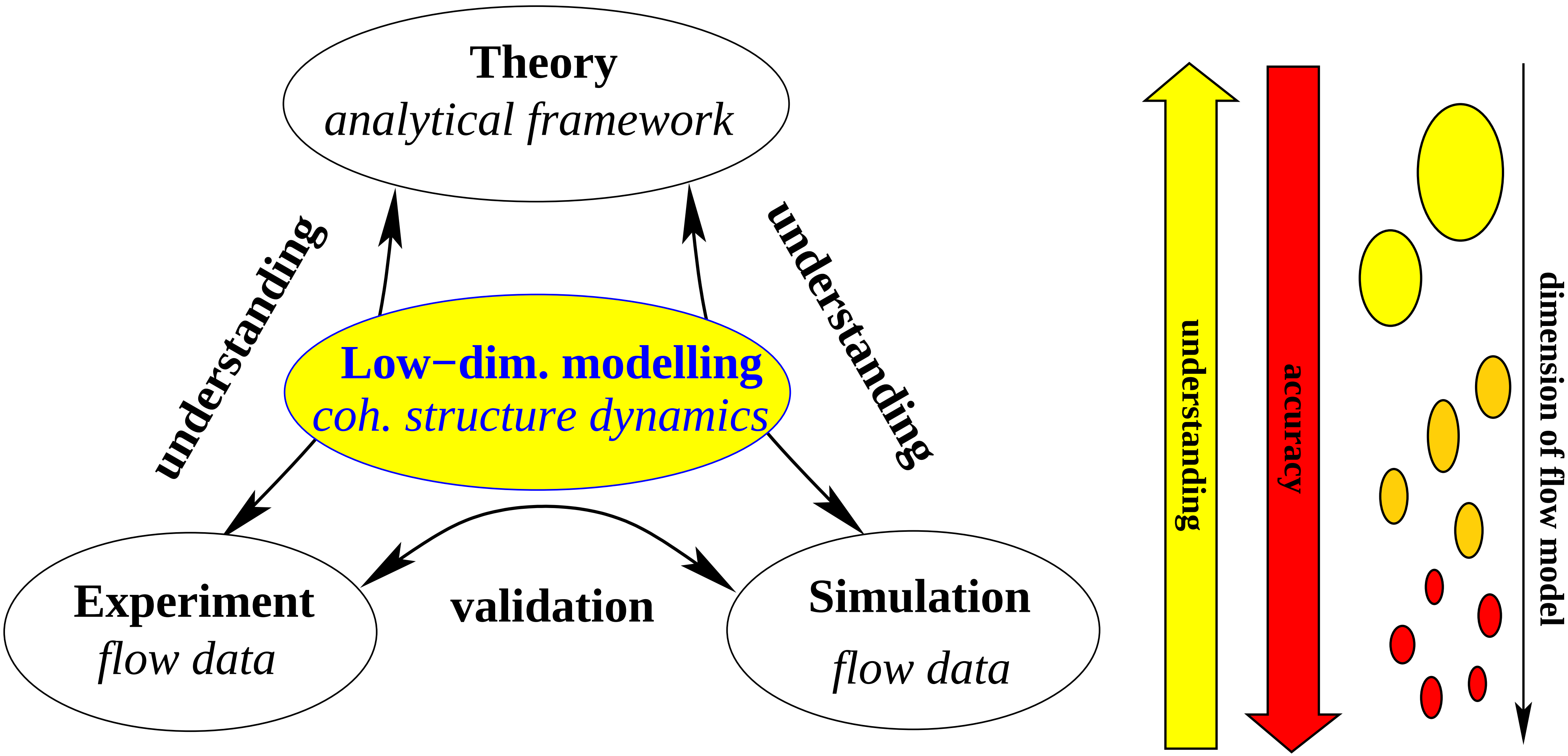
2. Motivation for reduced-order models

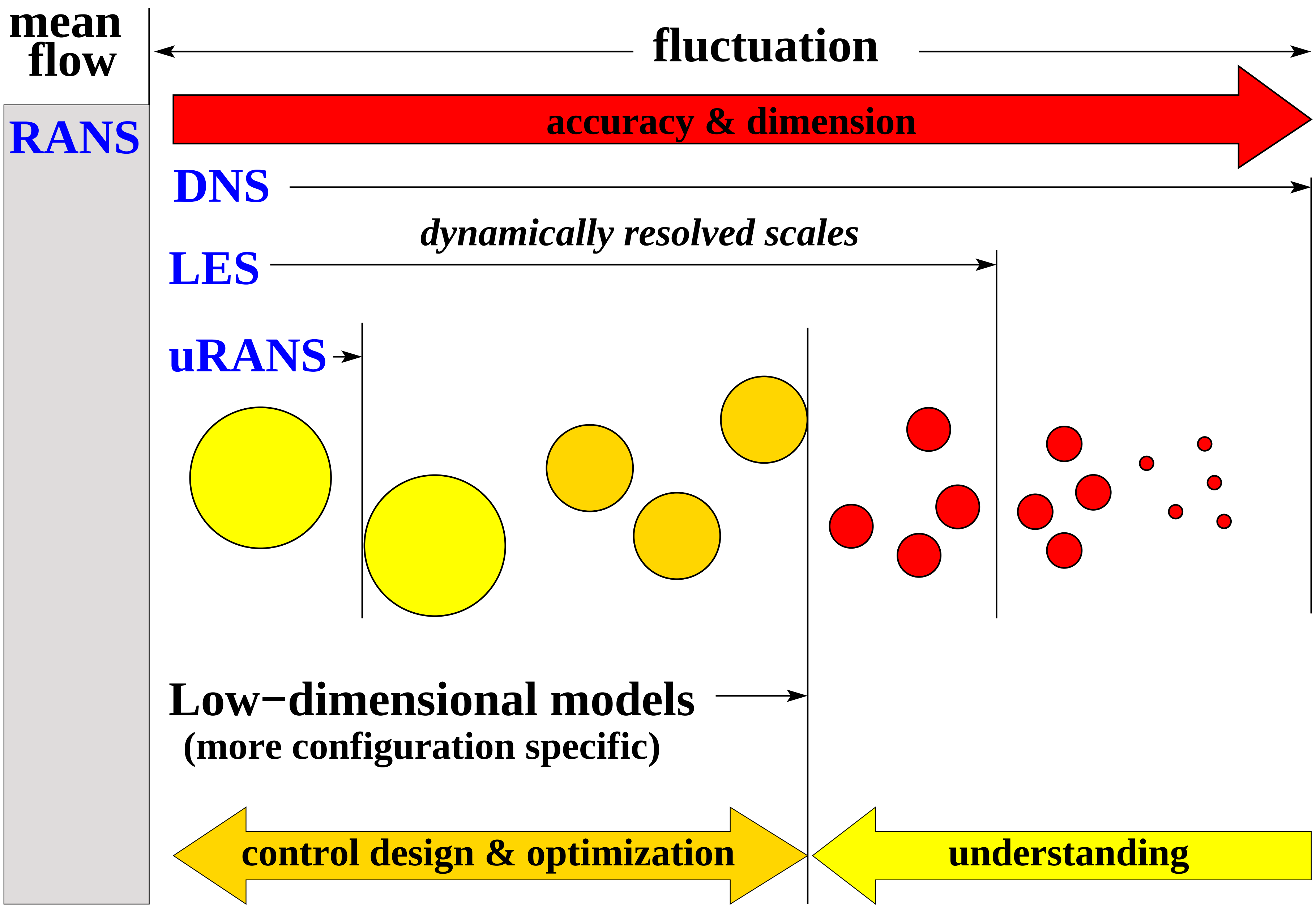
3. Examples of Galerkin models

4. Take home messages

Motivation: theoretical fluid dynamics



Motivation: computational fluid dynamics



Motivation: ecological & economical traffic

Modern traffic requires smart(er) flow control
- **increasing traffic** *(→ACARE vision 2020)*
- **increasing energy costs** *(oil price, ...)*
- **increasing ecological constraints** *(emissions, noise)*

already well investigated in 20th century fluid mechanics — aerodyn. design passive actuators — (over)optimised for one operating condition

↓

Hardware enablers: powerful, cheap & reliable actuators (and sensors) — **open-loop active control** — **applicable for off-design conditions**

↓

***Emerging* software enablers:**
- **reduced-order models**
- **control theory**

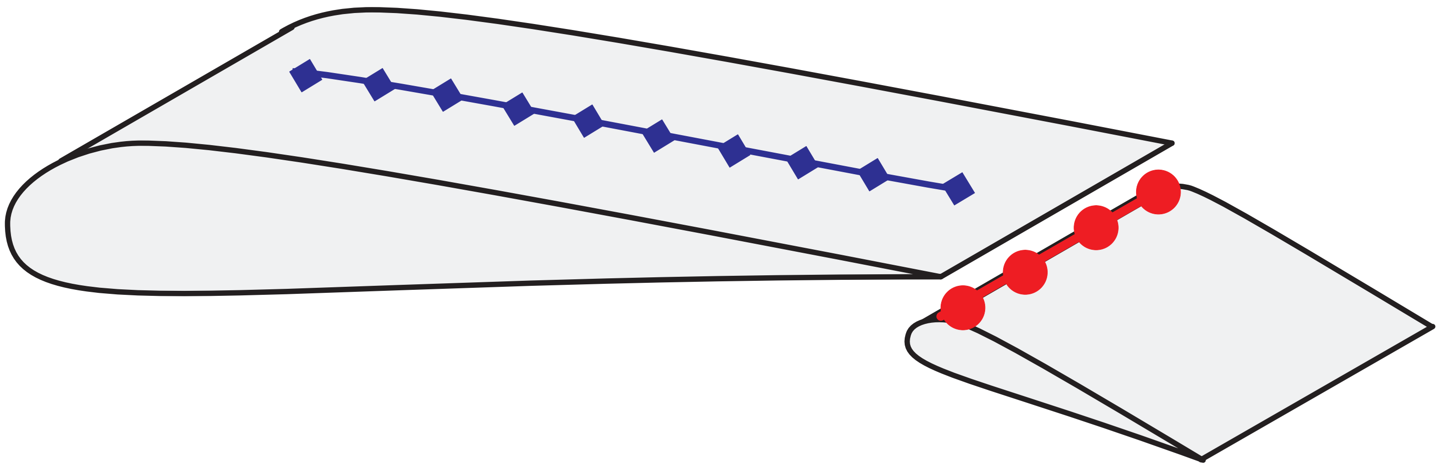
closed-loop active control

needed for
- **instability suppression**
- **disturbance compensation**
- **improved efficiency**

↓

21st century challenges of flow control in real-world applications:
- **control-oriented ROM handling complex flows** *(multi-scale, multi-physics)*
- **nonlinear control theory** *that respects the limitation of ROM*
- **implementation in experiment** *(online-capability, filtering, ...)*

Low-order modeling for flow control



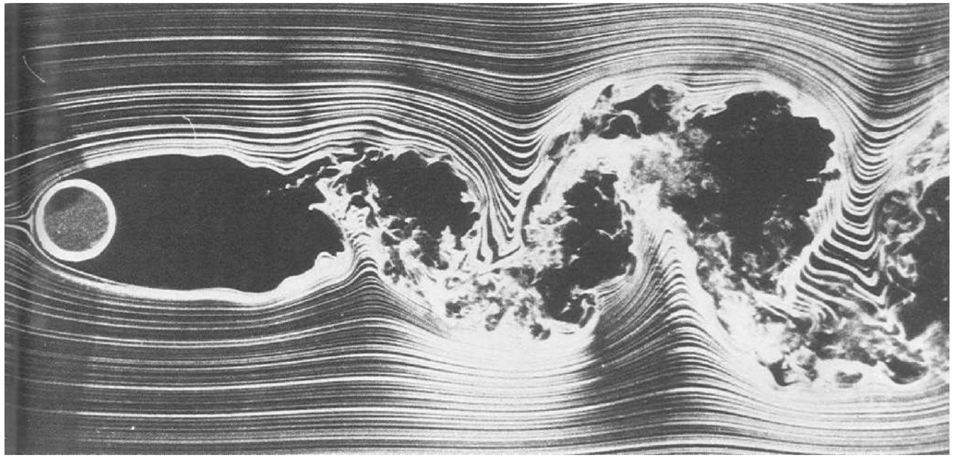
- Myriads of **actuation**- and **sensor**-opportunities:
  - • kind, • location, • amplitude and frequency range,
  - • control design
- No time for myriads of high-fidelity simulations.
- **Complementary low-dimensional models needed for exploration, optimization and control design.**

Path to low-dimensional models

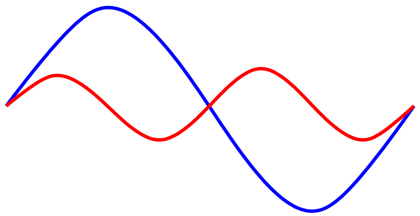
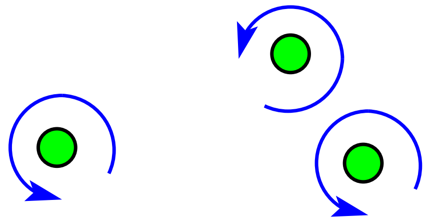
Low-dimensional coherent structures

Smoke visualization at $Re = 10000$

[van Dyke, Album of Fluid Motion]



Modelling approaches

	Eulerian view	Lagrangian view
coherent structures		
variable	velocity $\mathbf{u}$	vorticity ω
kinematics	$\mathbf{u} = \sum a_i(t)\mathbf{u}_i(\mathbf{x})$ Galerkin approximation	$\omega = \sum \Gamma_i \mathbf{\Omega}(\mathbf{x} - \mathbf{x}_i)$ vortex configuration
dynamics	$\frac{da_i}{dt} = f_i(a_1, \ldots)$ **Galerkin model**	$\frac{d\mathbf{x}_i}{dt} = \mathbf{u}(\{\Gamma_i, \mathbf{x}_i\})$ **vortex model**

Milestones of low-dimensional modeling

Leonardo da Vinci (1452–1519)

H.L.F. von Helmholtz (1821–1894)

Boris G. Galërkin (1871–1945)

Edward N. Lorenz (1917–2008)

- $\sim$ 1500 **da Vinci** $\in$ **visualization community**: First drawings of **coherent structures** (vortices) in the flow behind obstacles

— *Euler (1755), Navier (1822), Stokes (1845)* —

- 1869 **von Helmholtz**: Theoretical foundation of the **vortex methods** with Helmholtz vortex laws
- 1915 **Galërkin**: Pioneering work on the **Galerkin methods**
- 1963 **Lorenz**: 3-dim. model for Rayleigh-Benard convection which lead to the "**Lorenz attractor**"

History of vortex methods

∼1500	**Leonardo da Vinci** sketches coherent structures in wakes
1869	**Theoretical foundation** of the vortex methods with Helmholtz vortex laws
ca. 1900	Discussion of **N-vortex dynamics**
1911	**Vortex model** for stability investigation of wake (von Kárḿan)
1931	**First numerical simulation** of shear-layer [Rosenhead] (potential vortices)
1900-60	**Many vortex models** for wakes, shear-layers and jets
1962	**Desillusement:** Birkhoff observes numerical instability of point-vortex approximation (no convergence)
1981	**Mathematical foundation** of vortex methods, vortex cores, diffusion. Convergence proofs by Beale & Majda
1980+	**Numerical 3D vortex methods** (with turbulence models) [Leonard, Meiburg, ...]
1990+	**Generalizations of vortex methods to DNS:** compressibility, no-slip condition. [Ghoniem, Koumoutsakos, Soterios, Winckelmann, ...]

History of Galerkin models

Boris Grigoryevich Galërkin

(Russian engineer und applied mathematician):

1871	Birth (necessary prerequisite)
1899	PhD at the Technical Univ. of St. Petersburg
Then	Engineer at Kharkov Lokomotive Industry
1906/7	Imprisonment (Russian sabatical) because of his anti-zar views = begin of a long and fruitfull carrier
1909	Polytechnical Institute of St. Petersburg
1915	**Pioneering work on Galerkin method**
1920	Head of the Faculty of Applied Math
1945	Death

History of POD models for flow control:

1957	**E. Lorenz:** POD decomposition for flows
1967	**Lumley:** POD modes = coherent structures
1988	**Aubry, Holmes, Lumley & Stone:** POD model of turbulent boundary layer
1999	**Graham et al.:** First POD model for control
2000	**Hinze & Afanasiev:** POD model for optimal control
2003	**Gerhard et al.:** POD model for SISO control

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Galerkin model of 1D PDE

Initial boundary value problem $\mathcal{R} := \partial_t u + \partial_x u = 0$

Boundary condition: $u(0,t) = u(2\pi,t), \quad \forall t \geq 0$

Initial condition: $u(x,0) = \cos(x), \quad \forall x$

Galerkin approximation .... $u^{[2]}(x,t) := a_1(t) \underbrace{\cos(x)}_{:=u_1(x)} + a_2(t) \underbrace{\sin(x)}_{:=u_2(x)}$

Galerkin projection

$$0 \;=\; \left(u_1, \mathcal{R}\left[u^{[2]}\right]\right)_\Omega = \int_0^{2\pi} dx\, u_1 \left[\partial_t u^{[2]} + \partial_x u^{[2]}\right]$$

$$= \int_0^{2\pi} dx\, \cos x\, \{\cos x\, [\dot{a}_1 + a_2] + \sin x\, [\dot{a}_2 - a_1]\} \qquad \Rightarrow \dot{a}_1 = -a_2.$$

$$0 \;=\; \left(u_2, \mathcal{R}\left[u^{[2]}\right]\right)_\Omega = \int_0^{2\pi} dx\, u_2 \left[\partial_t u^{[2]} + \partial_x u^{[2]}\right] \qquad \Rightarrow \dot{a}_2 = +a_1$$

Galerkin system	Initial condition at $t=0$	**Galerkin solution**
$\dot{a}_1 = -a_2$	$a_1 = 1$	$a_1 = \cos t$
$\dot{a}_2 = +a_1$	$a_2 = 0$	$a_2 = \sin t$

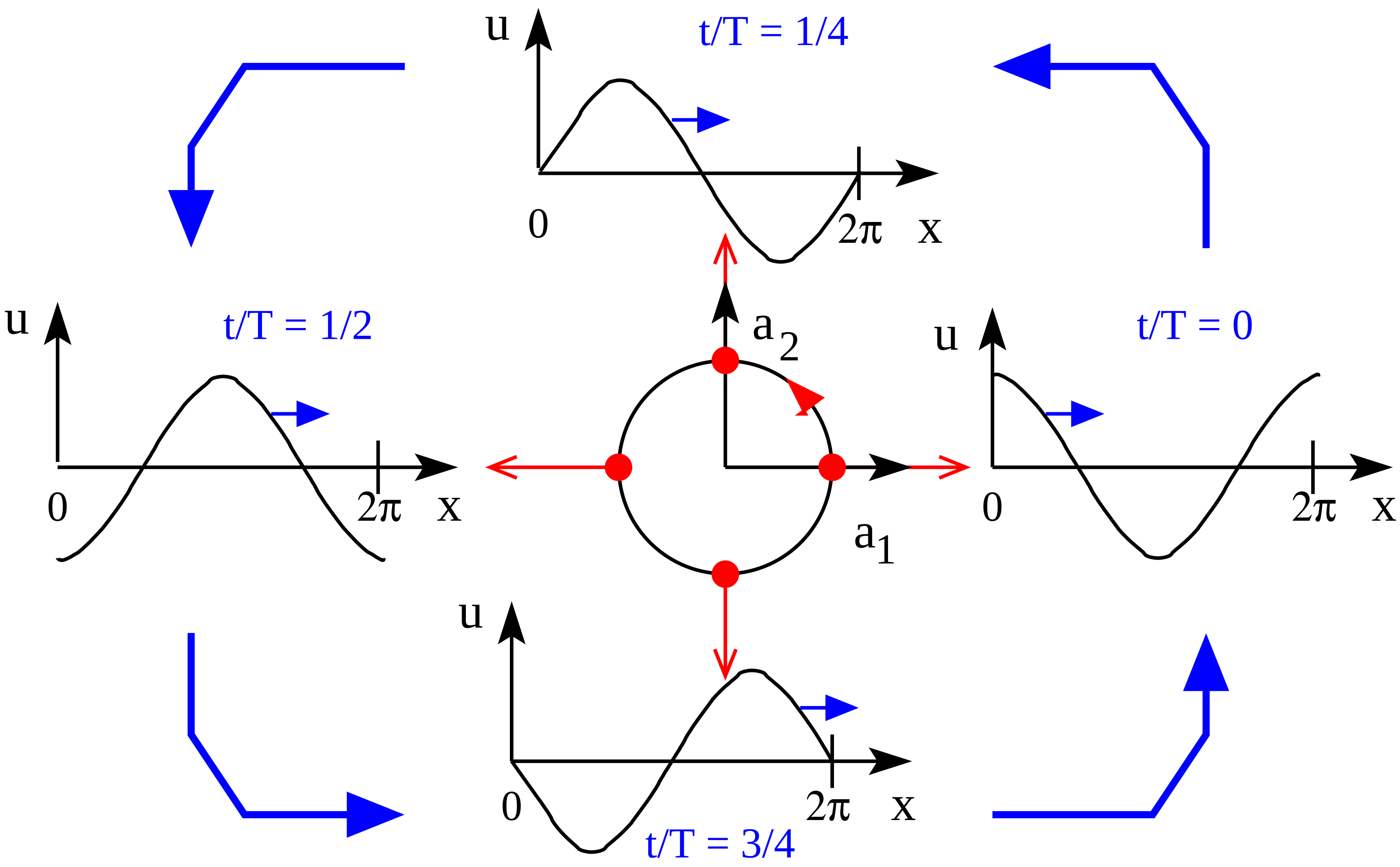
Galerkin solution fulfills exactly the IBVP

$$u^{[2]} := \cos t \cos x + \sin t \sin x = \cos(x - t)$$

Galerkin model of 1D PDE

Galerkin solution

$$u^{[2]} := \underbrace{\cos t}_{a_1} \underbrace{\cos x}_{u_1} + \underbrace{\sin t}_{a_2} \underbrace{\sin x}_{u_2} = \cos(x-t)$$



The art of Galerkin modeling

consists of making a good choice for its constitutive elements!

$$\mathbf{u}^{[0..N]} = \sum_{i=0}^{N} a_i \mathbf{u}_i, \qquad a_0 \equiv 1$$

$$\frac{d}{dt} a_i = \nu \sum_{j=0}^{N} l_{ij} a_j + \sum_{j,k=0}^{N} q_{ijk} a_j a_k$$

1. Basic mode $\mathbf{u}_0$:

Candidates:

- steady solution
- averaged solution
- mathematical flow

2. Hilbert space for $\mathbf{u}' = \mathbf{u} - \mathbf{u}_0$:

Candidates (solenoidal subsets):

- $\mathcal{L}^2(\Omega)$: $(\mathbf{u}, \mathbf{v})_\Omega = \int d\mathbf{x}\, \mathbf{u} \cdot \mathbf{v}$
- $\mathcal{L}^2_\sigma(\Omega)$: $(\mathbf{u}, \mathbf{v})_\Omega = \int d\mathbf{x}\, \sigma(\mathbf{x})\, \mathbf{u} \cdot \mathbf{v}$

3. Expansion modes $\mathbf{u}_i$:

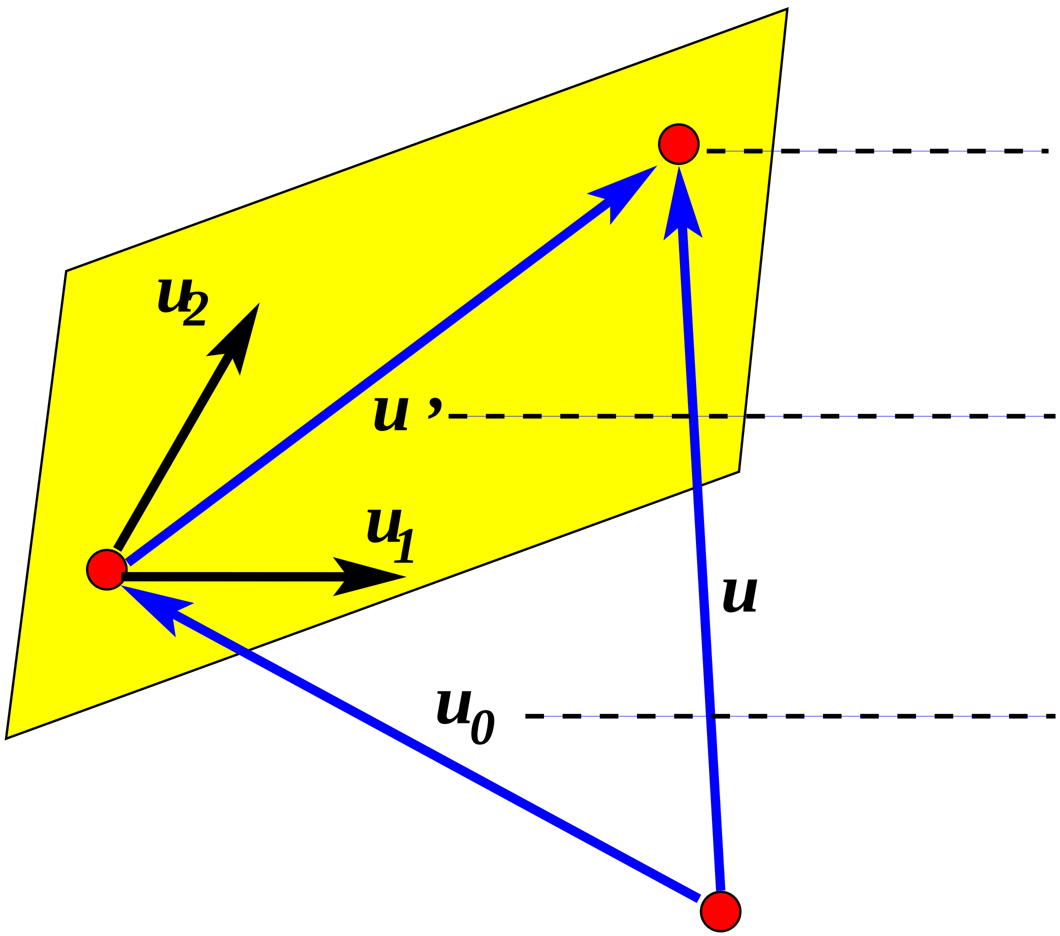
modes	BC	NSE	Solution
mathematical	X		
physical	X	X	
empirical	**X**	**X**	**X**

4. Traditional Galerkin projection on NSE:

$\left(\mathbf{u}_i, \mathcal{N}(\mathbf{u}^{[0..N]})\right)_\Omega = 0 \qquad \mathcal{N}$: Navier-Stokes residual

Low-dimensional Galerkin approximation

$$\mathbf{u}(\mathbf{x},t) = \mathbf{u}_0(\mathbf{x}) + \underbrace{\sum_{i=1}^{N} a_i(t)\,\mathbf{u}_i(\mathbf{x})}_{\mathbf{u}'}$$



Galerkin approximation

fulfills BC

fluctuation $\mathbf{u}'$

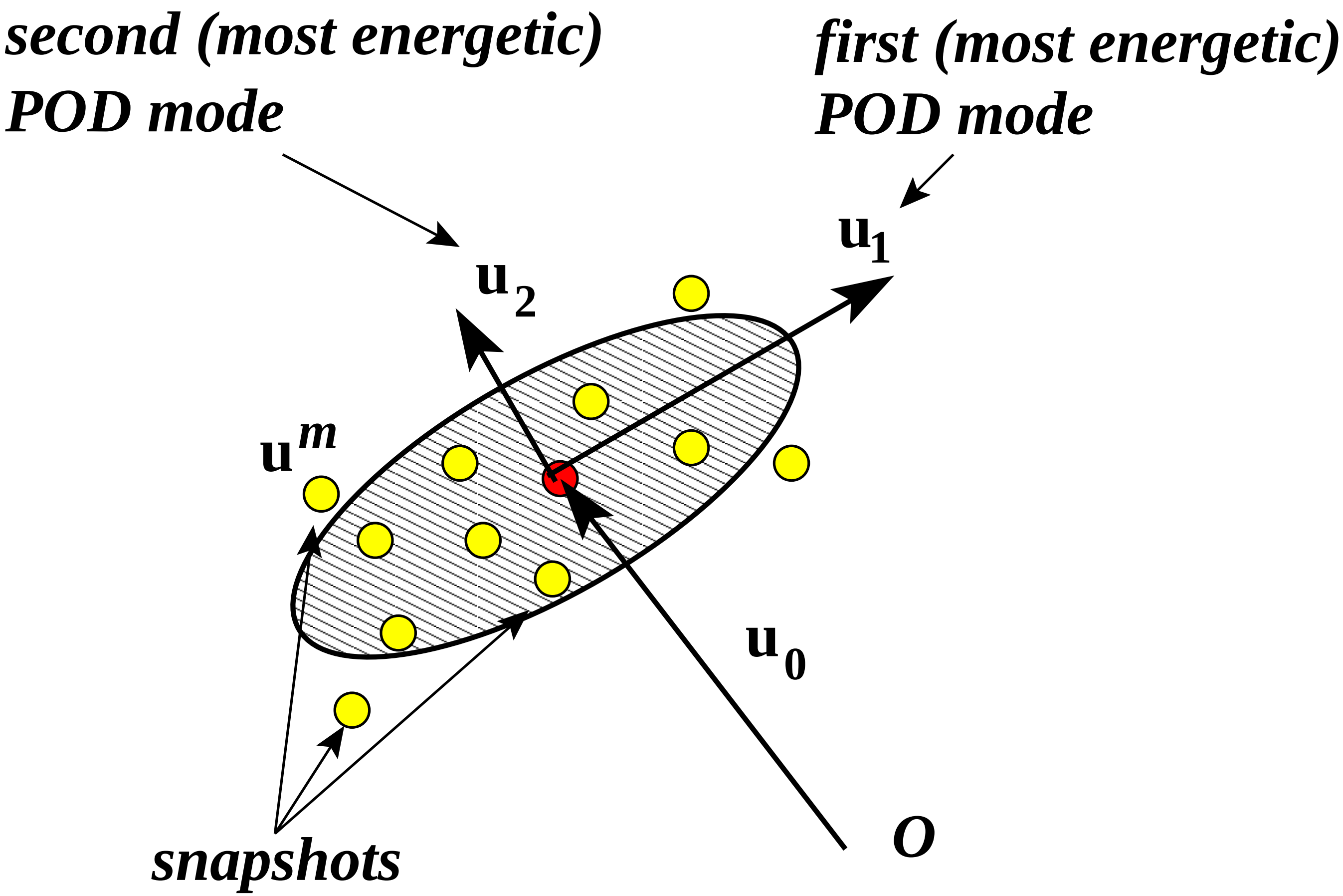
fulfills homogenized BC

basic mode

fulfills inhom. BC

'Least-dim.' (POD) Galerkin approximation

$$\mathbf{u}(\mathbf{x},t) = \mathbf{u}_0(\mathbf{x}) + \sum_{i=1}^{N} a_i(t)\,\mathbf{u}_i(\mathbf{x})$$



'Traditional' Galerkin method

— *Fletcher 1984 Computational Galerkin Methods, Springer* —

Galerkin method $\nu = 1/Re$

$$
\begin{array}{ccccccccc}
\mathbf{u} & \rightarrow & \partial_t \mathbf{u} & = \nu \triangle \mathbf{u} & & -\nabla(\mathbf{u}\mathbf{u}) & & -\nabla p & \\
\downarrow & & \downarrow & \downarrow & & \downarrow & & \downarrow & \\
\mathbf{u}^{[N]} = \sum_{i=0}^{N} a_i \mathbf{u}_i & \rightarrow & \frac{da_i}{dt} & = \nu \sum_{j=0}^{N} l_{ij} a_j & + & \sum_{j,k=0}^{N} (q^c_{ijk} & & + q^\pi_{ijk}) \, a_j \, a_k &
\end{array}
$$

Inner product . . . . $(\mathbf{u}, \mathbf{v})_\Omega := \int_\Omega dV \, \mathbf{u} \cdot \mathbf{v}$

Galerkin approximation with orthon. modes $(\mathbf{u}_i, \mathbf{u}_j)_\Omega = \delta_{ij}$

Galerkin projection exemplified for $\partial_t \mathbf{u}$

$$
\left(\mathbf{u}_i, \partial_t \mathbf{u}^{[N]}\right)_\Omega = \left(\mathbf{u}_i, \partial_t \left[\sum_{j=0}^{N} a_j \mathbf{u}_j\right]\right)_\Omega = \sum_{j=1}^{N} \frac{da_j}{dt} \underbrace{(\mathbf{u}_i, \mathbf{u}_j)_\Omega}_{\delta_{ij}} = \frac{d}{dt} a_i
$$

Minimal POD model

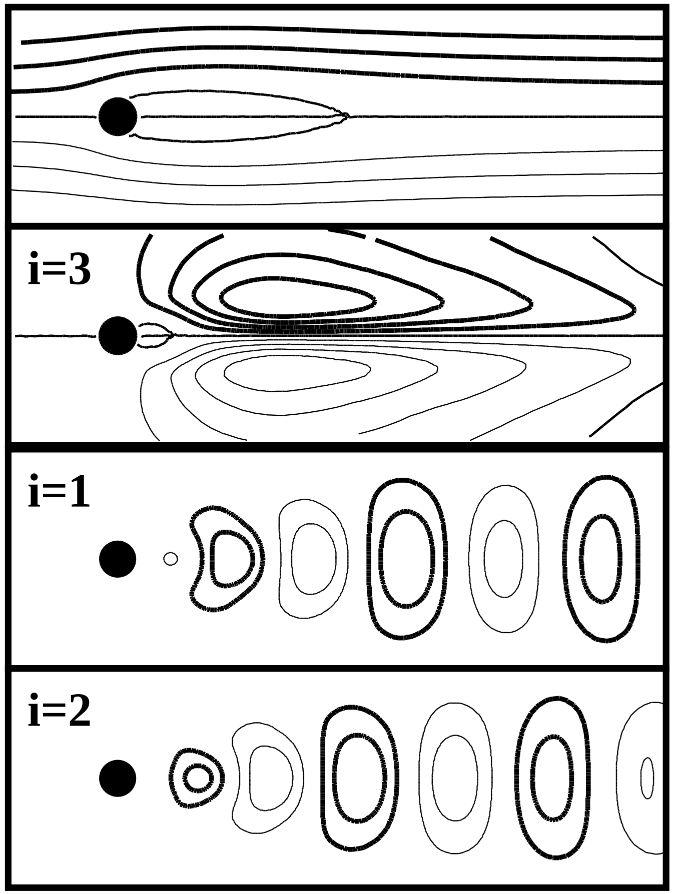
Noack, Afanasiev, Morzyński, Tadmor & Thiele (2003) JFM

$$\mathbf{u} \rightarrow \partial_t \mathbf{u} = \nu \triangle \mathbf{u} - \nabla(\mathbf{u}\mathbf{u}) - \nabla p$$

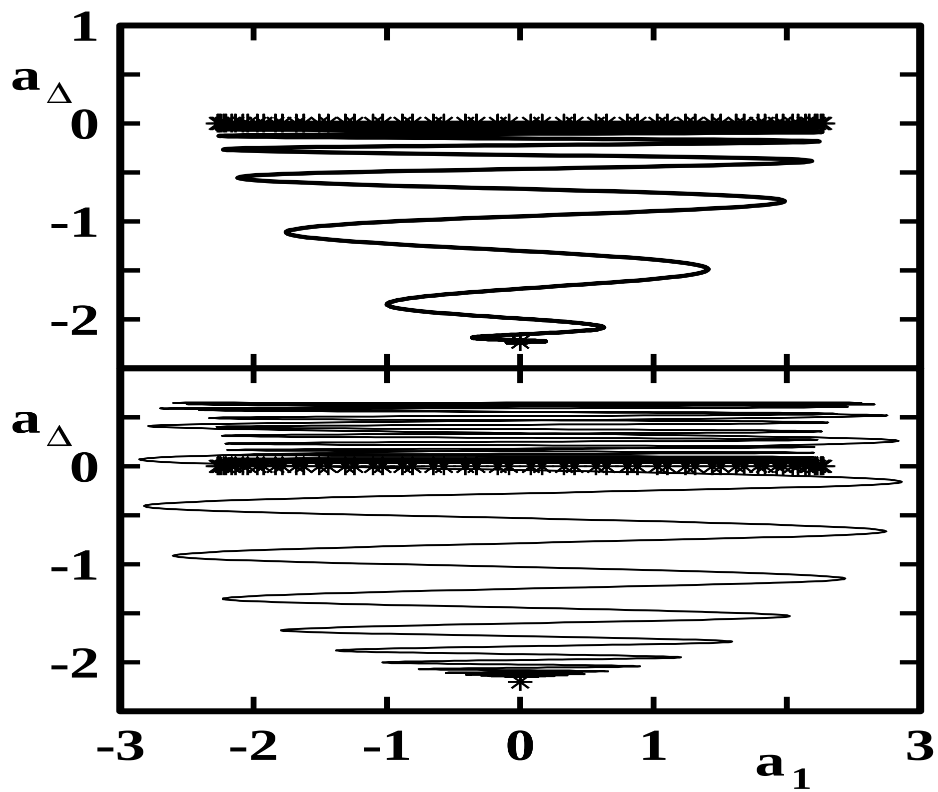
$$\downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow$$

$$\mathbf{u} = \sum_{i=0}^{3} a_i \mathbf{u}_i \rightarrow \frac{da_i}{dt} = \nu \sum_{j=0}^{3} l_{ij} a_j + \sum_{j,k=0}^{3} q_{ijk}\, a_j\, a_k$$

Modes



Transient dynamics



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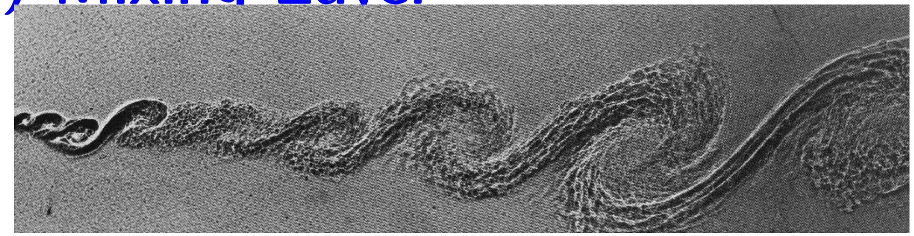
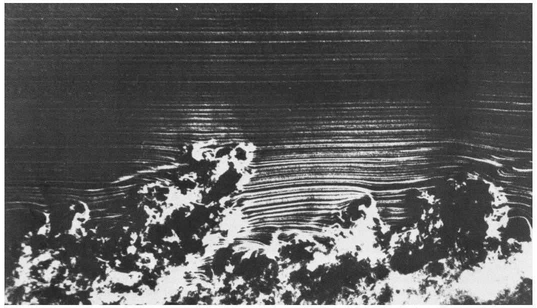
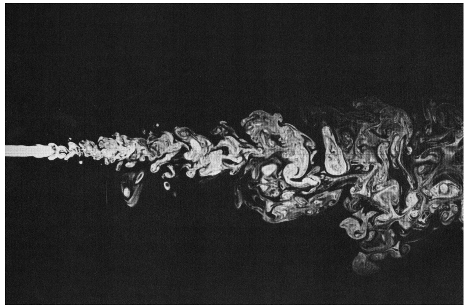
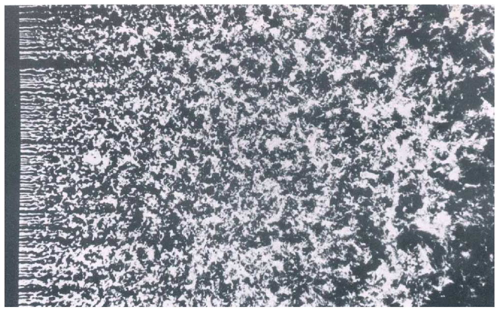
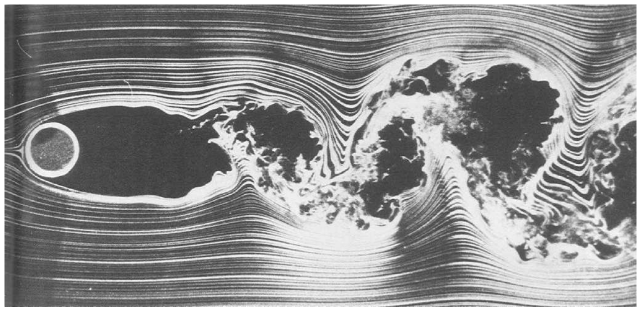
Conclusions

- Where there is light, there is shaddow.
  Where there are coherent structures, there is a ROM (eventually).
- Morever, the Galerkin method is a straightforward path (learnable art) to such ROM.
- ROM help physical understanding (bridge data-theory barrier).
- ROM = cheap alternative to CFD (narrow dynamic bandwidth).
- ROM = key for control design (to be shown by Gilead, Laurent & Rudi).

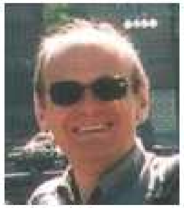
Göttingen shear flow classification

wall-bounded	free
1) Couette/channel flow 	3) Mixing Layer 
2) Boundary layer 	4) Jet 
6) Isotropic turbulence 	5) Wake 

Toolbox and configurations

our team

H.-C. Hege
ZIB, Berlin



M. Morzynski
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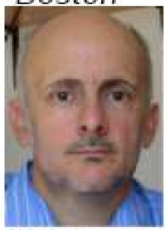


R. King
TU Berlin

our tools

data analysis
- feature extraction
- modal expansions

low order modelling
- Galerkin model
- vortex model
- black-box model

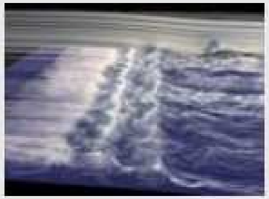
control design
- robust
- nonlinear

implementation
- filtering techniques
- online capability

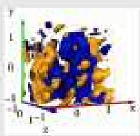
simple configurations

cylinder wake

mixing layer



backward facing step



homogeneous turbulence, ...

demonstrators funded by my DFG projects

car

airplane

aeroengine (compressor, combustor)

jet noise